

A Note on Genericity and Stability of Black Holes and Naked Singularities in Dust Collapse

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We comment here on the results in Ref [4] that showed naked singularities in dynamical gravitational collapse of inhomogeneous dust to be stable but non-generic. The definition of genericity used there is reconsidered. We point out that genericity in terms of an open set, with a positive measure defined suitably on the space of initial data, is physically more appropriate compared to the dynamical systems theory definition used in [4] which makes both black holes and naked singularities non-generic as collapse outcomes.

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In an early discussion, Hawking [1] emphasized the importance of the stability and genericity aspects for spacetime properties in gravitation theory. He pointed out that stable properties have a special significance in a physical theory when considering the correspondence between certain physical observations and a mathematical model. For general relativity the model is a spacetime, a four-dimensional manifold with a certain Lorentz metric, satisfying Einstein equations. The accuracy of real world observations is always limited by practical difficulties and by the uncertainty principle. Thus only properties of spacetime which are physically significant are those that are stable in an appropriate topology and unstable properties would not have physical relevance.

As the theory of gravitational collapse in general relativity evolved over a period of past four decades, it is known now that dynamical gravitational collapse of a massive matter cloud can end in either a black hole or a naked singularity final state, for spherical spacetimes with a variety of matter fields and also in many non-spherical models. Gravitational collapse has been studied by many authors in detail (see for example [2] and references therein). Existence of black hole (BH) or naked singularity (NS) as endstates of collapse is obtained depending on the choice of initial data from which the collapse evolves, as was shown by Joshi and Dwivedi (see for example [3]).

A natural question which then arises is, whether these occurrences and the collapse outcomes in terms of BH or NS final states are stable and generic with respect to the regular initial data on an initial spacelike

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surface from which the gravitational collapse develops.

From such a perspective, in the case of inhomogeneous dust, Saraykar and Ghate [4] showed that the occurrence of NS and BH is stable with respect to small variations in initial data functions in the sense that initial data set leading the collapse to NS (or BH) forms an open subset of the full initial data set under a suitably defined topology on the space of initial data. The authors assumed there the definition of genericity as given in the theory of dynamical systems (see e.g. [5]), and it was then argued that the NS occurrence is stable but not generic as per that definition. It is to be noted, however, that the occurrence of black holes also then turns out to be non-generic according to such a criterion. Therefore, the definition of genericity, as used in the dynamical systems studies would not be adequate to be used for discussing the gravitational collapse outcomes. An important point here is, work of different researchers over a period of last two decades has shown that the class of initial data set which leads to NS is disjoint and ‘fully separated’ from that which leads to BH. This is true for radiation collapse, inhomogeneous dust collapse and also for general type I matter fields ([2,4,6,7]).

Therefore neither black holes nor naked singularities can be dense in the full space of the initial data, and it is clear that a different criterion is necessary to access the genericity of these collapse final states. To examine this point, we consider here the situation of the inhomogeneous dust collapse following [3,4]: The spacetime metric in the case of inhomogeneous dust is given by

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2d\Omega^2, \quad (1)$$

and the energy momentum tensor and field equations are,

$$T^{ij} = \rho \delta^i_t \delta^j_t, \quad (2)$$

$$\rho(t, r) = \frac{F'}{R^2 R'} \quad (3)$$

$$\dot{R}^2 = \frac{F(r)}{R} + f(r) \quad (4)$$

where T^{ij} is the energy-momentum tensor, ρ is the total energy density and $F(r)$ and $f(r)$ are arbitrary functions of r . The dot denotes a derivative with respect to time, while a prime denotes a derivative with respect to r . Integration of equation (4) gives

$$t - t_0(R) = \frac{-R^{\frac{3}{2}} G\left(\frac{-fR}{F}\right)}{\sqrt{F}} \quad (5)$$

where the function $G(x)$ takes value $2/3$ at $x = 0$, and is expressed as inverse sine and inverse hyperbolic sine for other ranges of x (see e.g. [3,4] for exact expressions for $G(x)$). Following the root equation method

of [3], the condition for existence of a naked singularity or black hole is given in terms of the function $\Theta_u(r)$ described as follows ([4]):

$$\Theta_u(r) = \frac{1}{\sqrt{g}(3r^2g + r^3g')} \left[\frac{(\frac{Q'}{g} - \frac{g'}{g})}{\sqrt{1 + \frac{Q}{G}}} + (\frac{g'}{g} - \frac{3Q'}{2Q})G(\frac{-Q}{g}) \right] \quad (6)$$

where the mass function $F(r)$ and energy function $f(r)$ are written as

$$F(r) = r^3g(r) \quad (7)$$

and

$$f(r) = r^2Q(r) \quad (8)$$

Moreover $g(r)$ and $Q(r)$ are sufficiently smooth (at least C^1) functions satisfying regularity and energy conditions. It is assumed that $g(r)$ satisfies (i) $g(r) > 0$ and (ii) $rg'(r) + 3g > 0$. These are positivity of mass and energy conditions.

Then, if the value of the function $\Theta_u(r)$ at $r = 0$ is greater than α where,

$$\alpha = \frac{13}{3} + \frac{5}{2} \times \sqrt{3}, \quad (9)$$

then the tangent to a nonspacelike curve will be positive, *i.e.* future directed nonspacelike curves will reach the singularity in the past. In other words, singularity will be naked, not covered by an event horizon. If this condition is reversed, we get a black hole.

We now consider A to be the class of all continuous functions $A(r)$ defined on $[0, r_b]$. Consider a subclass of A , denoted by A_1 , consisting of functions $A(r)$, such that $A(0) > \alpha$. Now consider the equation

$$\Theta_u(r) = A(r), \quad (10)$$

regarded as differential equation in $Q(r)$ where $\Theta_u(r)$ is given by (6).

Then the following result was proved in [4] : Given a function $g(r)$ satisfying the above conditions, there are infinitely many choices of function $A(r)$ in the class A_1 such that for each such choice of $A(r)$, there exists a unique function $Q(r)$ such that the initial data $(g(r), Q(r))$ leads the collapse to a naked singularity. Thus, the conditions on $g(r)$ and $A(r)$ leading the collapse to NS are,

(i) $g(r) > 0$, (ii) $rg'(r) + 3g > 0$ and (iii) $A(0) > \alpha$.

Since the existence of $Q(r)$ is guaranteed by $g(r)$ and $A(r)$, we can consider the set N of all $(g(r), A(r))$ (instead of $(g(r), Q(r))$), satisfying the above conditions, as the set of initial data leading the collapse to NS. As proved in [4], this set forms an open subset of $G \times A$ where G is the space of all C^1 functions

defined on the interval $[0, r_b]$, and A is as above. We note that since the consideration of function $A(r)$ comes from equation (6) which contains $g(r)$, $Q(r)$ and their first derivatives, it is sufficient to assume that the functions $A(r)$ are continuous.

Similarly, the set B of all $(g(r), A(r))$ satisfying the conditions,

(i) $g(r) > 0$, (ii) $rg'(r) + 3g > 0$, and (iii) $A(0) < \alpha$,

will lead the collapse to a black hole, and similar arguments imply that B is also an open subset of $G \times A$. In this sense, both these BH and NS occurrences in collapse are stable. It is also clear that N and B are disjoint. Therefore, none of them can be dense in $G \times A$. Nevertheless, each of these sets are substantially big and it can be shown that they have a non-zero positive measure [8].

It is thus clear that if we follow strictly the dynamical systems definition of genericity, then both the outcomes of collapse, namely NS and BH would be non-generic. Thus, it is reasonable to argue that a change in the definition of genericity is desirable. This change is also justified by the work of other relativists who used the nomenclature 'generic' in the sense of 'abundance' or existence of an open set of non-zero measure consisting of initial data leading the collapse to BH or NS as in the case of scalar fields or for AdS models (see *e.g.* [9, 10] and references therein, but see also [11] where genericity is defined in terms of codimension). We note that in general it is clear from the definitions of 'stability' and 'genericity' used that these concepts depend upon the topologies under consideration. Thus a given property may be stable and generic in some topologies and not so in others. Which of the topologies is of physical interest will depend upon the nature of the property under consideration.

It follows that the dynamical systems definition of 'genericity' needs to be modified if we desire to have black holes as generic outcomes of gravitational collapse of dust. Dust collapse is clearly one of the most fundamental collapse scenarios, as the classic Oppenheimer-Snyder homogeneous dust collapse model is at the very foundation of the modern black hole physics and its astrophysical applications.

We could therefore formulate an appropriate and physically reasonable criterion of genericity for the dust collapse outcomes as follows:

We assume that the collapse begins with a regular initial data, with weak energy condition and other regularity conditions satisfied, *e.g.* that there are no shell crossings $R' = 0$ as the collapse evolves. The initial data, namely $F(r)$ and $f(r)$ (or $g(r)$ and $A(r)$) allow for the formation of both black holes and naked singularities, and we call each of these outcomes to be generic if the following conditions are satisfied:

(i) The set of initial data which evolves the collapse to naked singularity (or black hole) is an open subset of the full space of initial data.

(ii) If we impose a positive measure on the space of initial data, then the set of initial data with each of these outcomes should have a non-zero measure in the total space.

The first condition above means given an initial data point $F_1(r)$ and $f_1(r)$, which evolves the collapse to naked singularity (black hole), there should be an open neighborhood of this data point such that each initial data in this neighborhood also evolves to the same outcome. The second condition here means that each of these outcomes are substantially big in the full space of initial data.

We see from the consideration above that this holds true for dust collapse. In other words, an outcome of collapse, either in terms of a black hole or naked singularity is called ‘generic’ if there exists a subset or region of the initial data space that leads the collapse to such an outcome, and which has a positive measure. Then actually how ‘big’ such a region or the subspace of the initial data would be, depends on the collapse model being considered. For example, for the Vaidya radiation collapse, each of the regions going to BH or NS seem to be both finitely big and with a non-zero measure, and for dust collapse also they seem to have essentially equal ‘sizes’.

Thus, we observe the following: With the above definition of genericity, we find that both the outcomes of dust collapse, namely black holes and naked singularities are generic and also stable (see also [8] for a recent discussion on perfect fluid collapse). Since denseness depends upon the parent set chosen as well as the choice of topology, choosing the definition such as above looks physically reasonable. In other words, it is reasonable to argue that a change in the definition of genericity which will make both these collapse final states generic is desirable.

As we are aware, the concepts such as ‘stability’ and ‘genericity’ which are so important for physical considerations, are not well-defined in the Einstein gravitation theory, unlike the Newtonian case. This is of course the crux of the problems associated with any precise formulation of the cosmic censorship conjecture. It is therefore clear that a deeper, more detailed and if possible case by case consideration of collapse models may help understand these aspects better. In this spirit, we believe the consideration above provides useful insights on cosmic censorship, which continues to be one of the most outstanding and important problems in gravitation theory today. The stability and genericity for collapse outcomes in the context of a general gravitational collapse will be discussed elsewhere.

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